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STABILITY OF A SEMI-INFINITE STRIP IN THERMOELASTIC CONTACT WITH A RIGID WALL

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Abstract—When heat is conducted across an interface between two dissimilar materials, thermoelastic distortion affects the contact pressure distribution. The existence of a pressure-sensitive thermal contact resistance at the interface can cause such systems to be unstable in the steady state. The stability of a semi-infinite strip in contact with a rigid wall is investigated using linear perturbation methods. The strip is assumed to be in uniform contact with the wall in the steady-state, with one-dimensional heat conduction along the strip and across the interface, where there is a pressure-dependent thermal contact resistance. Possible perturbations are expressed in the form of an eigenfunction series, using the Papkovich–Fadle eigenfunctions for the strip and related eigenfunctions for the thermoelastic particular solution. Selection of perturbations that can grow exponentially in time lead to an eigenvalue problem for the coefficients of the series.

Results show that stability is governed by a symmetric perturbation with approximately sinusoidal form across the width of the strip. The stability boundary is quite well approximated by a simplified analysis assuming an exactly sinusoidal perturbation, particularly when the dimensionless thermal contact resistance in the steady state is small.

1. INTRODUCTION

When heat is conducted across an interface between two dissimilar materials, thermoelastic distortion affects the contact pressure distribution. The existence of a pressure-sensitive thermal contact resistance at the interface can cause such systems to be unstable in the steady state.

Early investigations of thermoelastic contact stability were restricted to one-dimensional systems such as a rod contacting a rigid wall (Barber *et al.*, 1980) or axisymmetric concentric cylinders of dissimilar materials (Barber, 1986). For these systems, conditions can be found for which the steady state is unique, in which case it is always stable. However, under certain conditions, multiple steady-state solutions are found, some of which are stable and other unstable. In such cases, we anticipate that whatever initial condition is chosen for the system, it will eventually gravitate to one of the stable steady states.

Barber and Zhang (1988) investigated the more complex one-dimensional system of two contacting rods of dissimilar materials and found that in this case steady-state solutions can be unstable even when they are unique. The transient behavior in such cases was investigated using the finite difference method and the results demonstrate that the system tends to a steady oscillatory state in which the contact pressure varies periodically with time, possibly with periods of separation.

The simplest two-dimensional problem is that of two elastic half-planes in contact at a common interface. Barber (1987) examined the stability of this system by a linear perturbation method. The assumed perturbation involved a sinusoidal variation in temperature and stress in the direction parallel to the interface. As in the one-dimensional case, uniqueness and stability criteria were found to coincide when one of the materials was taken to be rigid, but when both half-planes were assumed to be deformable, cases could be

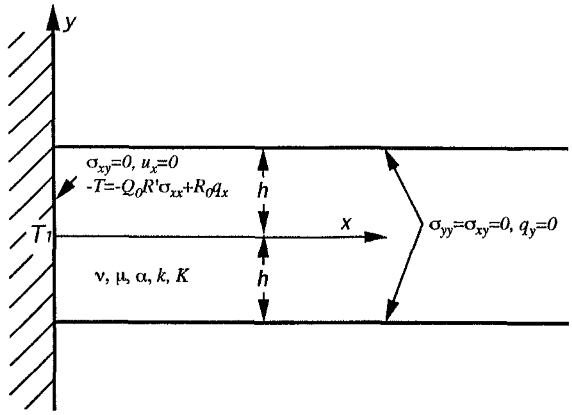

Fig. 1. A semi-infinite strip contacting a rigid wall.

found for which the steady state could be unique but unstable. In a subsequent investigation, Zhang and Barber (1990) showed that the stability criterion depended on the values of three dimensionless ratios of the bimaterial properties and that a substantial number of practical material combinations would exhibit this more complex behavior.

Yeo and Barber (1991) investigated the effect of a finite geometry on the thermoelastic stability for a system consisting of a layer on the half-plane. One feature found in this problem is that, unlike the two half-plane problem, the stability criterion is not necessarily determined by the perturbational temperature with the largest wavelength: there is a preferred wavelength for which the system is most likely to be unstable for specific material combinations when the heat flows into the layer. These conditions are associated with complex roots and typically involve perturbations whose wavelength are of the order $2\pi h$, where h is the layer thickness.

The two-dimensional systems so far studied are of infinite dimension in the direction parallel with the contact plane and it follows that if the perturbation is expressed as a Fourier series or transform in this dimension, stability will be determined by that sinusoidal term which first achieves a growth rate with positive real part. When the body is finite in the contact plane dimension, this simplification is lost, since there is then no reason to expect the dominant perturbation to be sinusoidal.

In this paper, we consider the simplest problem of this type, in which the end face of a semi-infinite strip is pressed against a hot rigid plane.

2. STATEMENT OF THE PROBLEM

The system under consideration is shown in Fig. 1. The semi-infinite strip Ω: $\{x \geqslant 0, |y| \leqslant h\}$ has insulated, traction-free edges at $y = \pm h$ and is pressed against a frictionless rigid plane surface at $x = 0$. Plane-strain conditions are assumed with respect to the direction z. The rigid plane is maintained at uniform temperature and heat flows to the strip across a thermal contact resistance $R(P)$ that is a function of the local contact pressure P. We consider the stability of the simple one-dimensional steady-state solution, in which the contact pressure is uniform and equal to P_0, the appropriate thermal resistance being $R(P_0) = R_0$. The stress field will then be one of uniaxial compression and the temperature field will be one-dimensional, corresponding to a uniform heat flux Q_0 in the x-direction.

As in previous studies, we investigate the conditions under which a small perturbation from this steady state can grow exponentially in time. The perturbed quantities must satisfy the following boundary conditions:

- Since the wall is frictionless and rigid, the perturbed shear traction and normal displacement must be zero at the contact plane, i.e.

$$\left.\begin{aligned}\sigma_{xy} &= 0\\ u_x &= 0\end{aligned}\right\} \quad \text{at } x = 0. \tag{1}$$

- The edges $y = \pm h$ are traction-free and insulated, so

$$\left.\begin{aligned}\sigma_{yy} &= 0\\ \sigma_{xy} &= 0\end{aligned}\right\} \quad \text{at } y = \pm h \tag{2}$$

and

$$q_y = -K\frac{\partial T}{\partial y} = 0 \quad \text{at } y = \pm h, \tag{3}$$

where K is the thermal conductivity of the material.
- The local heat flux Q, temperature difference ΔT and thermal contact resistance R at the interface are related by the equation

$$\Delta T = QR. \tag{4}$$

Linear perturbation of this relation [see for example Yeo and Barber (1991), eqn (39)] about the uniform steady-state values yields the boundary condition

$$-T(0, y) = -Q_0 R' \sigma_{xx}(0, y) + R_0 q_x(0, y), \tag{5}$$

where T, q_x, σ_{xx} are the perturbations in temperature, heat flux and stress, respectively, Q_0, R_0 are the steady-state heat flux into the strip and contact resistance, and $R' = \mathrm{d}R/\mathrm{d}P$ is the derivative of the contact resistance with respect to variations in contact pressure P about the mean value P_0.

3. THE TEMPERATURE PERTURBATION

The perturbation in the temperature field $T(x, y, t)$ must satisfy the thermal boundary conditions (3) and be sufficiently general to describe all possible perturbations in the strip. However, we first examine temperature perturbations of the restricted separated-variable form

$$T(x, y, t) = f(y)\, \mathrm{e}^{ax}\, \mathrm{e}^{bt}. \tag{6}$$

We shall later generate the general solution by superposition of such terms in the form of an eigenfunction expansion.

The temperature must satisfy the heat conduction equation

$$\nabla^2 T = \frac{1}{k}\frac{\partial T}{\partial t}, \tag{7}$$

where k is the thermal diffusivity of the material. Substituting eqn (6) into eqn (7) and cancelling the exponential terms, we find that $f(y)$ must satisfy the ordinary differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} + \left(a^2 - \frac{b}{k}\right) f = 0, \tag{8}$$

whose solution is

$$f(y) = A_s(a,b)\cos(\gamma y) + A_a(a,b)\sin(\gamma y), \tag{9}$$

where

$$\gamma^2 = a^2 - \frac{b}{k}. \tag{10}$$

Notice that the function $f(y)$ contains one symmetric and one antisymmetric term, multiplied by arbitrary constants A_s, A_a respectively. In view of the symmetry of the geometry about $y = 0$, the complete solution can be partitioned into symmetric and antisymmetric fields, which will generally have different sets of eigenfunctions for the exponential growth rate b. The derivations for the symmetric and antisymmetric solutions are essentially similar, and only the symmetric case is developed in detail here.

Taking the symmetric term from eqn (9) and substituting into the thermal boundary condition (3) at $y = \pm h$, we obtain

$$A_s(a,b)\sin(\gamma h) = 0, \tag{11}$$

which defines a non-trivial temperature field if and only if $\gamma h = n\pi$, where n is an integer. A more general symmetric solution for the perturbed temperature field can then be constructed by superposition in the form

$$T(x,y,t) = \sum_{n=1}^{\infty} A_n(b)\cos(\gamma_n y)\,\mathrm{e}^{a_n x}\,\mathrm{e}^{bt}, \tag{12}$$

where

$$\gamma_n = \frac{n\pi}{h}, \tag{13}$$

and a_n is the root with negative real part of the double-valued quantities

$$a_n = (\gamma_n^2 + b/k)^{1/2}, \tag{14}$$

since the perturbation must decay away from the contact plane.† If the temperature at $x = 0$ were any prescribed symmetric zero-average function of the form $f(y)\exp(bt)$, substitution in eqn (12) and Fourier inversion would permit the constants A_n to be determined. Thus, eqn (12) is a complete representation for perturbations of this exponential form.

4. THE THERMOELASTIC PROBLEM

To complete the solution, we need to determine the contact pressure distribution at $x = 0$ resulting from the thermoelastic distortion associated with the temperature distribution (12) under the boundary conditions (1, 2) and substitute the resulting expressions into the thermal contact resistance equation (5), which will then define an eigenvalue problem for b, the components of the corresponding eigenfunction being the constants A_n. We anticipate an infinite sequence of eigenvalues for b, and furthermore that an arbitrary initial perturbation could be described as a series of the associated eigenfunctions. It follows that the system will be stable if and only if *all* the eigenfunctions for b have negative real parts and that the stability boundary—i.e. the critical value of the steady-state heat flux above which a perturbation will grow without limit—is that at which any eigenvalue first crosses the imaginary axis.

† Notice that we have excluded the case $n = 0$ in the series (12), since the corresponding fields are independent of y and are therefore unable to satisfy the condition that the perturbed tractions on $x = 0$ be self-equilibrated.

Previous investigations have shown that unstable growth is always associated with real eigenvalues when only one material is in contact with a rigid surface, whereas complex eigenvalues can arise when two deformable materials are in contact.† It follows that the stability boundary is defined by the lowest heat flux having $b = 0$ as an eigenvalue and some simplification can be achieved by restricting attention to this case in the following analysis. It then follows that

$$a_n = -\gamma_n = -n\pi/h \tag{15}$$

and the temperature perturbation of eqn (12) reduces to

$$T = \sum_{n=1}^{\infty} A_n \cos(\gamma_n y) e^{-\gamma_n x}. \tag{16}$$

4.1. *Thermal distortion due to free thermal expansion*

It is convenient to separate the thermoelastic problem into two parts—we first determine the distortion of the strip associated with free thermal expansion, all the surfaces being assumed traction-free, and then superpose an isothermal solution to determine the contact pressure distribution when this thermally distorted strip is pressed into frictionless contact with the rigid plane at $x = 0$.

For $b = 0$, the perturbed temperature field (16) is independent of time and hence satisfies the Laplace equation. It then follows (Dundurs, 1974; Barber, 1992) that there will be no internal stresses if the boundaries are traction-free and also that the thermoelastically-induced curvature of the boundaries will be proportional to the local heat flux. In particular, we have

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{\alpha(1+\nu)}{K} q_x, \tag{17}$$

where $\delta = \alpha(1+\nu)/K$ is known as the thermal distortivity and α, ν, K are the thermal expansion coefficient constant, Poisson's ratio and thermal conductivity, respectively, for the material.

The heat flux into the body at $x = 0$ corresponding to the temperature distribution (16) is

$$q_x = -K\frac{\partial T}{\partial x} = K \sum_{n=1}^{\infty} A_n \gamma_n \cos(\gamma_n y); \quad x = 0. \tag{18}$$

Substituting this result into eqn (17) and integrating, we obtain

$$u_x|_{x=0} = u_0 = -\alpha(1+\nu) \sum_{n=1}^{\infty} A_n \frac{\cos(\gamma_n y)}{\gamma_n}, \tag{19}$$

omitting a rigid body displacement.

4.2. *Determination of contact pressure*

We next determine the self-equilibrated perturbation in contact pressure required to retain frictionless contact between the thermally-distorted strip of Section 4.1 and the rigid plane. This pressure distribution must clearly produce surface displacements equal and opposite to u_0 on the end $x = 0$ and hence defines the isothermal end-strip problem, where a semi-infinite strip $\Omega: \{x \geqslant 0, |y| \leqslant h\}$ is subjected to boundary conditions

† Parallel numerical studies, to be published separately, show that this is also the case for the present geometry, but the authors have been unable as yet to prove a general theorem to this effect.

$$\left.\begin{aligned} u_x = -u_0 = \alpha(1+\nu)\sum_{n=1}^{\infty} A_n \frac{\cos(\gamma_n y)}{\gamma_n} \\ \sigma_{xy} = 0 \end{aligned}\right\} \quad \text{at } x = 0 \tag{20}$$

and

$$\left.\begin{aligned} \sigma_{yy} = 0 \\ \sigma_{xy} = 0 \end{aligned}\right\} \quad \text{at } y = \pm h. \tag{21}$$

The problem of a semi-infinite strip $\Omega : \{x \geqslant 0, |y| \leqslant h\}$ with homogeneous boundary conditions on the edges $y = \pm h$ was solved using the Airy stress function and the resulting eigenfunction expansion is associated with the names of Papkovitch (1940) and Fadle (1941).

The problem can also be solved using the harmonic strain potentials A and D of Green and Zerna (1954). The appropriate stress and displacement components then take the form

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial x^2} - 2\nu \frac{\partial \varphi}{\partial y} + y \frac{\partial^2 \varphi}{\partial x^2} \tag{22}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial y^2} + y \frac{\partial^2 \varphi}{\partial y^2} - 2(1-\nu) \frac{\partial \varphi}{\partial y} \tag{23}$$

$$\sigma_{xy} = \frac{\partial^2 \phi}{\partial x \, \partial y} + y \frac{\partial^2 \varphi}{\partial x \, \partial y} - (1-2\nu) \frac{\partial \varphi}{\partial x} \tag{24}$$

$$2\mu u_x = \frac{\partial \phi}{\partial x} + y \frac{\partial \varphi}{\partial x} \tag{25}$$

$$2\mu u_y = \frac{\partial \phi}{\partial y} + y \frac{\partial \varphi}{\partial y} - (2-4\nu)\varphi, \tag{26}$$

where ϕ, φ, are harmonic functions of x, y, and μ is the shear modulus.

We first consider particular potential functions ϕ, φ in the separated variable form

$$f(x,y) = g(y)\, e^{\lambda x}, \tag{27}$$

in which case, the condition $\nabla^2 f = 0$ leads to the ordinary differential equation

$$\frac{\partial^2 g(y)}{\partial y^2} + \lambda^2 g(y) = 0. \tag{28}$$

The solution of this equation is

$$g(y) = C(\lambda)\cos(\lambda y) + D(\lambda)\sin(\lambda y), \tag{29}$$

where $C(\lambda)$ and $D(\lambda)$ are arbitrary constants.

Substitution into eqns (22–27) shows that a state of stress symmetric about $y = 0$ is obtained if we take only the cosine term in the expression for ϕ and only the sine term in φ, i.e.

$$\phi(x,y) = C(\lambda)\cos(\lambda y)e^{\lambda x}; \quad \varphi(x,y) = D(\lambda)\sin(\lambda y)\,e^{\lambda x}. \tag{30}$$

Substituting eqns (30) into (23, 24) and applying the traction-free edge conditions (2) results in the equations

$$-\lambda^2\cos(\lambda h)C(\lambda) - \{\lambda^2 h\sin(\lambda h) + 2(1-\nu)\lambda\cos(\lambda h)\}D(\lambda) = 0 \tag{31}$$

$$\lambda^2\sin(\lambda h)C(\lambda) - \{\lambda^2 h\cos(\lambda h) - (1-2\nu)\lambda\sin(\lambda h)\}D(\lambda) = 0, \tag{32}$$

which have a non-trivial solution if and only if λ is a root of the characteristic equation

$$2\lambda h + \sin(2\lambda h) = 0. \tag{33}$$

Equation (33) has a simple zero at $\lambda = 0$, and in addition a conjugate pair of zeros $\lambda_m, \bar{\lambda}_m$ in each interval $(m-\frac{1}{2})\pi < \mathrm{Re}(\lambda_m h) < (m-\frac{1}{4})\pi$. Also $-\lambda_m$, $-\bar{\lambda}_m$ are zeros. At these eigenvalues for λ, eqns (31, 32) are not linearly independent, both reducing to

$$D(\lambda_m) = -\frac{\kappa_m}{h}C(\lambda_m), \tag{34}$$

where

$$\kappa_m \equiv \kappa(\lambda_m h) = \frac{\lambda_m h}{2(1-\nu) - \sin^2(\lambda_m h)}, \tag{35}$$

which, through eqns (30), defines the eigenfunction of the system.

A more general solution to the strip problem can now be constructed in the form of an eigenfunction series. Since the perturbational temperature and the resulting stresses must decay as $x \to \infty$, we include only those eigenvalues with negative real part. Adopting the numbering convension that $\lambda_1, \lambda_2, \ldots$ are the zeros in the second quadrant, we then write

$$\lambda_{-m} = \bar{\lambda}_m, \quad m = 1, 2, 3, \ldots \tag{36}$$

The potential functions can then be written

$$\phi(x,y) = \sum_{m=-\infty}^{+\infty} C_m\cos(\lambda_m y)\,e^{\lambda_m x}; \quad \varphi(x,y) = -\frac{\kappa_m}{h}\sum_{m=-\infty}^{+\infty} C_m\sin(\lambda_m y)\,e^{\lambda_m x}, \tag{37}$$

using eqns (30, 34) where we write $C_m \equiv C(\lambda_m)$. The required traction and displacement components at $x = 0$ are

$$\sigma_{xx} = \sum_{m=-\infty}^{+\infty}\left(\frac{C_m}{h^2}\right)\psi_m^{(\sigma)}, \tag{38}$$

$$\sigma_{xy} = \sum_{m=-\infty}^{+\infty}\left(\frac{C_m}{h^2}\right)\psi_m^{(\tau)}, \tag{39}$$

$$2\mu u_x = \sum_{m=-\infty}^{+\infty}\left(\frac{C_m}{h}\right)\psi_m^{(u)}, \tag{40}$$

$$2\mu u_y = \sum_{m=-\infty}^{+\infty}\left(\frac{C_m}{h}\right)\psi_m^{(v)}, \tag{41}$$

from eqns (37, 22, 24–26), where

$$\psi_m^{(\sigma)}(\lambda_m y) = (\lambda_m h)^2 \cos(\lambda_m y) + \kappa_m\{2\nu\lambda_m h \cos(\lambda_m y) - \lambda_m^2 hy \sin(\lambda_m y)\} \tag{42}$$

$$\psi_m^{(\tau)}(\lambda_m y) = -(\lambda_m h)^2 \sin(\lambda_m y) - \kappa_m\{\lambda_m^2 hy \cos(\lambda_m y) - (1-2\nu)\lambda_m h \sin(\lambda_m y)\} \tag{43}$$

$$\psi_m^{(u)}(\lambda_m y) = \lambda_m h \cos(\lambda_m y) - \kappa_m \lambda_m y \sin(\lambda_m y) \tag{44}$$

$$\psi_m^{(v)}(\lambda_m y) = -\lambda_m h \sin(\lambda_m y) - \kappa_m\{\lambda_m y \cos(\lambda_m y) - (3-4\nu)\sin(\lambda_m y)\}. \tag{45}$$

4.2.1. *Use of the biorthogonality condition.* The constants $\{C_m\}$ can be determined from the boundary data at $x = 0$ (Spence, 1982, 1983; Gregory, 1979, 1980a,b). In general, these boundary conditions in combination with eqns (38–41) will lead to an infinite set of linear equations for the $\{C_m\}$, but there are some problems where they can be found explicitly, using a biorthogonality condition. Gregory (1983) (see also Johnson and Little, 1965) obtained the biorthogonality relation satisfied by the elastodynamic (or elastostatic) eigenfunctions for a cylindrical rod of general cross-section by using the elastic reciprocal theorem, and the elastic symmetry of the cylinder about planes perpendicular to its generators. In the two-dimensional problem of the strip, the biorthogonality condition takes the form

$$\int_0^h (u_x^l \sigma_{xx}^m - \sigma_{xy}^l u_y^m)\,\mathrm{d}y = 0 \quad l \neq m, \tag{46}$$

where $\{u_x^l, \sigma_{xy}^l\}$, $\{u_y^m, \sigma_{xx}^m\}$ are the eigenfunctions corresponding to the eigenvalues λ_l, λ_m, respectively.

In the terminology of the present paper, the biorthogonality condition takes the form

$$\int_0^h (\psi_l^{(u)}\psi_m^{(\sigma)} - \psi_l^{(\tau)}\psi_m^{(v)})\,\mathrm{d}y = \begin{cases} \dfrac{2(1-\nu)(\lambda_m h)^3 h \cos^2(\lambda_m h)}{(2(1-\nu)-\sin^2(\lambda_m h))^2} & l = m \\ 0 & l \neq m \end{cases} \tag{47}$$

as can be verified by integration, using eqns (33, 35, 42–45).

To apply this relation to the case where u_x, σ_{xy} are prescribed at the boundary, we multiply both sides by C_l and sum on l in the range $(-\infty, +\infty)$, obtaining

$$\sum_{l=-\infty}^{+\infty} C_l \int_0^h (\psi_l^{(u)}\psi_m^{(\sigma)} - \psi_l^{(\tau)}\psi_m^{(v)})\,\mathrm{d}y = \frac{2(1-\nu)C_m(\lambda_m h)^3 h \cos^2(\lambda_m h)}{(2(1-\nu)-\sin^2(\lambda_m h))^2}. \tag{48}$$

The summation on the left-hand side of this equation can then be taken under the integral sign and the summation performed, using eqns (39, 40), with the result

$$\int_0^h (2\mu h u_x(0,y)\psi_m^{(\sigma)} - h^2\sigma_{xy}(0,y)\psi_m^{(v)})\,\mathrm{d}y = \frac{2(1-\nu)C_m(\lambda_m h)^3 h \cos^2(\lambda_m h)}{(2(1-\nu)-\sin^2(\lambda_m h))^2}, \tag{49}$$

giving the coefficient C_m explicitly in terms of the prescribed end data u_x, σ_{xy}.

For the special case where the end data are given by eqn (20), we obtain

$$C_m = 2\mu\alpha(1+\nu)h^2 \sum_{n=1}^{\infty} C_{mn} A_n, \tag{50}$$

where

$$C_{mn} = \frac{(2(1-\nu)-\sin^2(\lambda_m h))^2}{2(1-\nu)\gamma_n h^2(\lambda_m h)^3\cos^2(\lambda_m h)}\int_0^h \cos(\gamma_n y)\psi_m^{(\sigma)}\,dy. \tag{51}$$

The normal contact traction at x = 0 is then obtained from (38,50) as

$$\sigma_{xx} = 2\mu\alpha(1+\nu)\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\psi_m^{(\sigma)}C_{mn}A_n. \tag{52}$$

4.3. *Contact resistance equation*

It remains to satisfy the boundary condition (5), which is the linearly perturbed form of the relation between heat flux, temperature difference, thermal contact resistance and contact pressure at the interface $x = 0$.

Substitution of eqns (16,18,52) into eqn (5) yields a set of homogeneous equations for the unknowns A_n, in which the steady-state heat flux, Q_0, functions as an eigenvalue. It is convenient to truncate the series at $n = N$ for A_n and at $2M$ terms in the Papkovich–Fadle expansion (52), in anticipation of numerical solution. The resulting equation takes the form

$$\sum_{n=1}^{N} R_0K\gamma_n\cos(\gamma_n y)A_n - Q_0R'2\mu\alpha(1+\nu)\sum_{n=1}^{N}\sum_{m=-M}^{M}\psi_m^{(\sigma)}C_{mn}A_n = -\sum_{n=1}^{N}\cos(\gamma_n y)A_n, \tag{53}$$

which must be satisfied for $0 < y < h$.

To reduce the number of parameters we define dimensionless quantities

$$\Gamma_n = \gamma_n h;\quad \Lambda_n = \lambda_n h;\quad Y = y/h. \tag{54}$$

Then $\psi_m^{(\sigma)}(\lambda_m y), C_{mn}$ in eqn (53) can be rewritten in terms of these quantities, the resulting expression being denoted by $\Psi_m^{(\sigma)}$ and C^*_{mn}, respectively, i.e.

$$\psi_m^{(\sigma)}(\lambda_m y) = \Lambda_m^2\cos(\Lambda_m Y) + \kappa(\Lambda_m)\{2\nu\Lambda_m\cos(\Lambda_m Y) - \Lambda_m^2 Y\sin(\Lambda_m Y)\} \equiv \Psi_m^{(\sigma)}(Y) \tag{55}$$

and

$$C_{mn} = \frac{[2(1-\nu)-\sin^2(\Lambda_m)]^2}{2(1-\nu)\Gamma_n\Lambda_m^3\cos^2(\Lambda_m)}\int_0^1 (\cos(\Gamma_n Y)\Psi_m^{(\sigma)})\,dY \equiv C^*_{mn}. \tag{56}$$

The use of the dimensionless quantities

$$Q^* = -Q_0R'2\mu\alpha(1+\nu);\quad R^* = \frac{R_0K}{h}, \tag{57}$$

then enables us to write eqn (53) as

$$\sum_{n=1}^{N}(R^*\Gamma_n+1)\cos(\Gamma_n Y)A_n = Q^*\sum_{n=1}^{N}\sum_{m=-M}^{M}\Psi_m^{(\sigma)}C^*_{mn}A_n \quad 0 < Y < 1, \tag{58}$$

where $\Gamma_n = n\pi$ and Λ_m is obtained from the equation

$$2\Lambda_m + \sin 2\Lambda_m = 0. \tag{59}$$

5. NUMERICAL PROCEDURE

The roots of eqn (59) can be found numerically by an iterative algorithm, given initial values for Λ_n, which are not too far from the correct values. Suitable initial values are provided by the asymptotic solution of Spence (1983) as

$$\Lambda_n = (n-\tfrac{1}{4})\pi + \frac{i}{2}\ln(4n\pi). \tag{60}$$

The truncation of the series in eqn (58) means that we can only hope to satisfy it in an approximate sense. Two essentially different methods of approximation could be considered:

(1) *Direct collocation*, in which the equation is satisfied exactly at a suitable set of collocation points $\{Y_i\}$ in the interval (0,1).
(2) *Galerkin method*, in which we require the error to be orthogonal to N linearly independent weighting functions W_i over the interval (0,1).

Both methods lead to a general eigenvalue problem with matrix size of $N \times N$. In this study, the Galerkin method was used, since it showed better numerical convergence.

From a numerical point of view, a good choice of weighting functions is one which makes the resulting two matrices diagonally dominant, hence making the eigenvalues—here Q^*—converge with increasing order of truncation.

The functions on both sides of eqn (58) involve trigonometric functions (albeit multiplied by linear functions in some cases), the wavelength of which are of the same order. Hence it is natural to choose as weighting functions

$$W_i = \cos(\Gamma_i Y), \tag{61}$$

resulting in the generalized eigenvalue problem

$$\mathbf{LA} = Q^*\mathbf{HA}, \tag{62}$$

where

$$L_{in} = (R^*\Gamma_n + 1)\int_0^1 (\cos(\Gamma_i Y)\cos(\Gamma_i Y))\,\mathrm{d}Y \tag{63}$$

and

$$H_{in} = \int_0^1 \left(\cos(\Gamma_i Y)\sum_{m=-M}^{M} \Psi_m^{(\sigma)}(Y)C_{mn}^*\right)\mathrm{d}Y$$
$$= \sum_{m=-M}^{M} C_{mn}^* \int_0^1 (\cos(\Gamma_i Y)\Psi_m^{(\sigma)}(Y))\,\mathrm{d}Y, \tag{64}$$

and $\mathbf{A}$ is a column matrix composed of the A_n. Equation (62) has an asymmetric matrix $\mathbf{H}$ and hence may have real and/or complex eigenvalues. However, only the real eigenvalues are physically meaningful for the dimensionless steady-state heat flux Q^* and complex eigenvalues are therefore disregarded.

6. NUMERICAL RESULTS

An essentially similar analysis can be performed for the anti-symmetric modes. Numerical results for the stability boundary for the first three eigenvalues of each mode are presented in Fig. 2, where the series in eqn (53) have been truncated at $N = 20$ and $M = N/2$. The only parameter in the dimensionless formulation is Poisson's ratio ν which was taken to be 1/3.

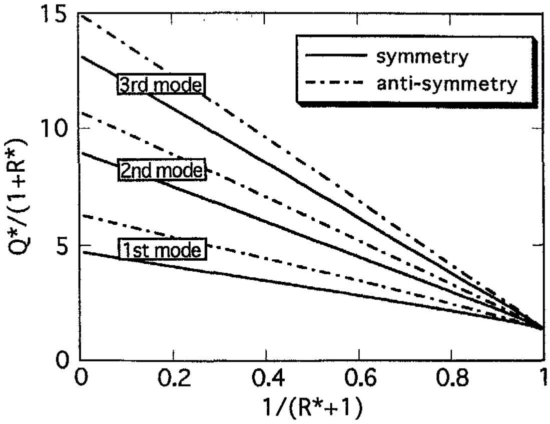

Fig. 2. Critical heat flux as a function of dimensionless contact resistance, $\nu = 1/3$.

The stability of the system is determined by the lowest of the curves in Fig. 2, which corresponds to the first eigenvalue of the symmetric problem. The corresponding mode shapes (eigenfunctions) for the heat flux at $x = 0$ are shown in Figs 3 and 4 for the symmetric and antisymmetric case respectively, with $R^* = 1$. The eigenfunctions are approximately sinusoidal in the y-direction and must have zero average value. Thus, the 'wavelength' of the nth symmetric eigenfunction is approximately $2h/n$ and that of the corresponding antisymmetric eigenfunction is $2h/(n+1/2)$. The longest wavelength corresponds to the first mode of the symmetric solution, agreeing with the previous results for the systems with pure sinusoidal eigenfunctions [4], which show that longer wavelengths become unstable at lower heat fluxes.

The limit in which the dimensionless resistance $R^* \to \infty$ is mathematically equivalent to the limit $h \to 0$, as can be seen from eqn (57). In this case, the "wavelength" of the first eigenfunction is necessarily very short and consequently the heat flux Q^* required for instability is very high. Notice that $Q^*/(1+R^*)$ rather than Q^* is plotted as ordinate in Fig. 2, since this enables the whole range of results to be plotted compactly. The numerical results suggest that $Q^*/(1+R^*)$ tends to a finite limit as $R^* \to \infty$ and hence that $Q^* \sim R^*$ in this limit.

The opposite limit where $h \to \infty$ and $R^* \to 0$ is equivalent to the problem of a half-plane in contact with a wall, treated by Barber (1987). The present results are in very good agreement with the analytical results of that paper.

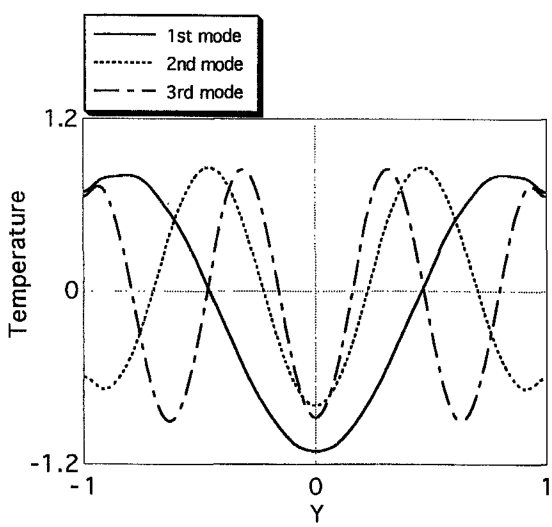

Fig. 3. Temperature perturbation at $x = 0$ (symmetric modes) $\nu = 1/3$, $R^* = 1.0$.

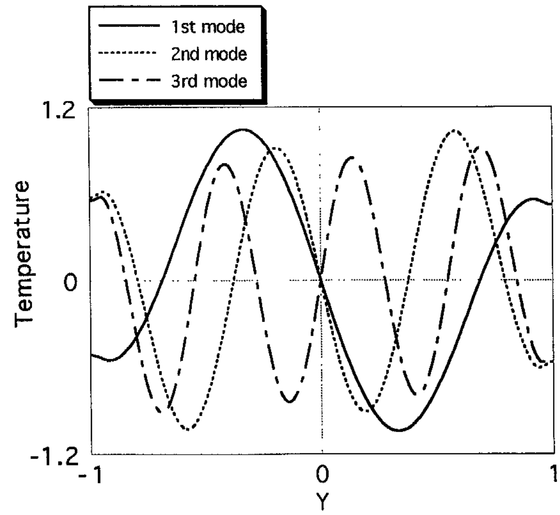

Fig. 4. Temperature perturbation at $x = 0$ (antisymmetric modes) $\nu = 1/3$, $R^* = 1.0$.

As indicated above, Barber's analytical solution for the half-plane can be used to obtain an approximate solution for the strip problem, based on the assumption that the eigenfunction will be sinusoidal across the thickness direction with wavelength $2h$. The sinusoidal approximation (dotted line) is compared with the present exact results (solid line) in Fig. 5. The sinusoidal approximation has been used, for example, to estimate the stability criterion for the related problem of frictionally excited thermoelastic instability (TEI) for an automotive brake pad of finite length (Dow and Burton, 1972; Lee and Barber, 1993). The present results show that, in all cases, the sinusoidal approximation predicts a lower stability boundary (i.e. a lower value of Q^*) than the exact solution, which means that it is a safe design approximation.

It is interesting to note that the curves in Fig. 5 are closely approximated by straight lines. This is also found to be the case for other values of ν. The present solution can be approximated by the equation

$$\frac{Q^*}{1+R^*} = 2(1-\nu)\left\{3.88\left(1-\frac{1}{1+R^*}\right)+\frac{1}{1+R^*}\right\}, \tag{65}$$

whilst the approximate solution based on a sinusoidal eigenfunction predicts

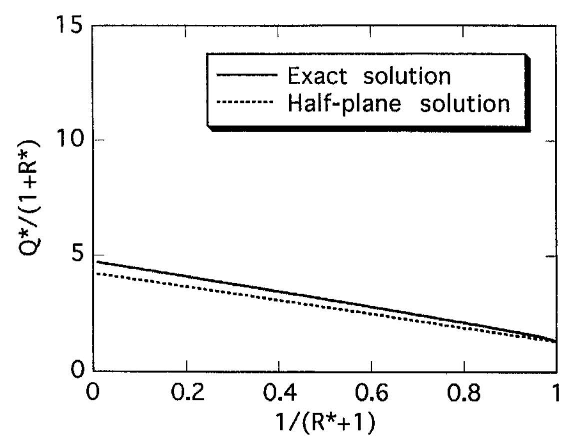

Fig. 5. Exact and approximate solutions for the stability boundary, $\nu = 1/3$.

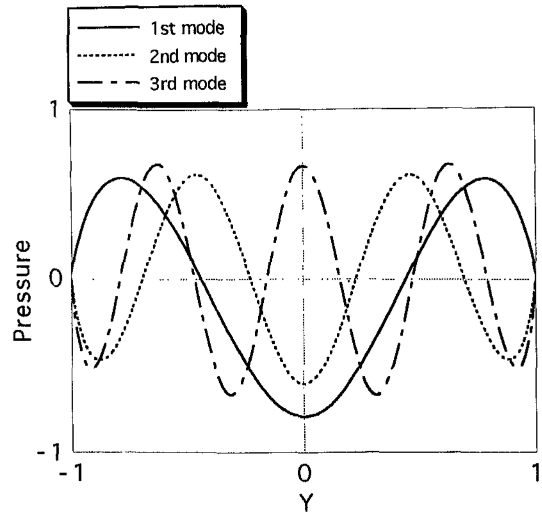

Fig. 6. Pressure perturbation (symmetric modes), $\nu = 1/3$, $R^* = 1.0$.

$$\frac{Q^*}{1+R^*} = 2(1-\nu)\left\{\pi\left(1-\frac{1}{1+R^*}\right)+\frac{1}{1+R^*}\right\}. \tag{66}$$

Figures 6 and 7 show the perturbed pressure distribution along the contact plane when $R^* = 1$. Since these are the eigenfunctions of the system, the vertical scale in these figures is arbitrary. Some irregularity in the pressure distribution can be observed for the anti-symmetric mode with $N = 20$, indicating that the numerical scheme converges more slowly for this mode. This phenomenon disappeared with increasing number of terms. However, the stability boundary calculated with $N = 20$ is still good enough, since the stability boundary is governed by the symmetric mode. This is confirmed by Fig. 8, which shows the convergence of the stability boundary with increasing N for $R^* = 1$. The results show excellent convergence and suggest that $N = 20$ is amply sufficient to ensure 0.1% accuracy in Q^*.

7. CONCLUSIONS

This paper presents the first solution to a thermoelastic contact problem in which the eigenfunctions are truly two-dimensional. Previous solutions have involved infinite domains

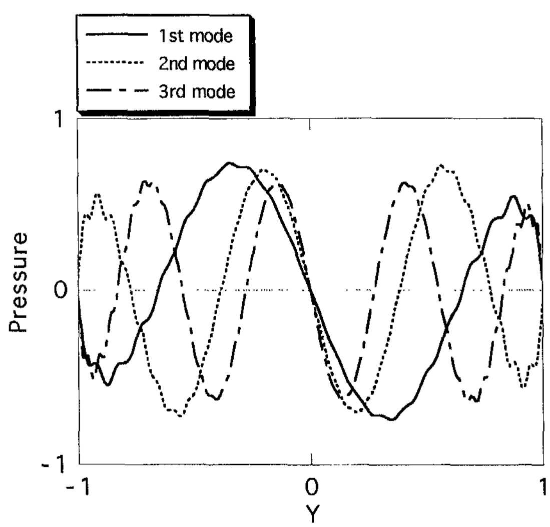

Fig. 7. Pressure perturbation (antisymmetric modes), $\nu = 1/3$, $R^* = 1.0$.

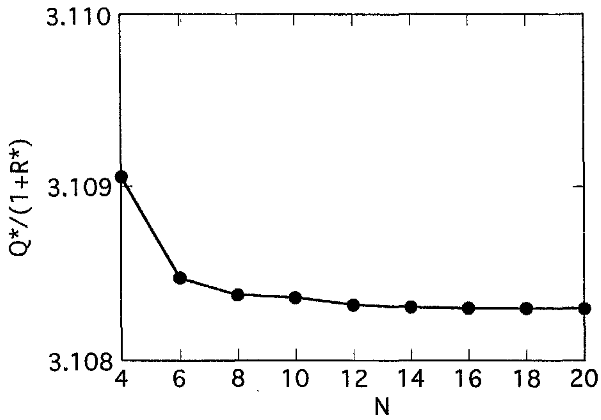

Fig. 8. Convergence of the critical heat flux with increasing N. $\nu = 1/3$, $R^* = 1.0$.

or periodic boundary conditions and therefore permitted the sinusoidal form of the function in one dimension to be inferred *a priori*. The stability boundary is determined by the finite geometry, which therefore must be adequately described if quantitative stability information is to be extracted for practical systems. Previous more practical analyses have tended to approximate the behavior of finite systems using the closest sinusoidal form. In the present problem this is equivalent to relating the strip under consideration to the problem with periodic boundary conditions formed by wrapping the strip into a thin-walled cylinder and joining the edges. The predictions of this approximation prove to be quite close to the exact solution for the strip, which is an encouraging result for design purposes.

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